Hence, $h_t \circ p$ is a homotopy between $p \circ F_0 = p \circ \mathrm{id}_X = p = \mathrm{id}_{X/A} \circ p$ and $p \circ F_1 = p \circ (q \circ p) = (p \circ q) \circ p$, so h_t is a homotopy between $\mathrm{id}_{X/A}$ and $p \circ q$.

Thus, p and q are mutually homotopy inverse, which completes the proof.

Corollary 2. *If (X, A) is a CW pair, then $X/A \sim X \cup CA$, where CA is a cone over A.*

Proof. $X/A = (X \cup CA)/CA \sim X \cup CA$. The latter follows from Corollary 1 applied to the CW complex $X \cup CA$ and its contractible CW subcomplex CA.

Remark. Both propositions may be regarded not as corollaries from Borsuk's theorem but as independent theorems, only the assumption of (X, A) being a CW pair should be replaced, in the first case, by the assumption that (X, A) is a Borsuk pair, and in the second case, by the assumption that $(X \cup CA, CA)$ is a Borsuk pair.

5.7 The Cellular Approximation Theorem

Theorem. *Every continuous map of one CW complex into another CW complex is homotopic to a cellular map.*

We will prove this theorem in the following, *relative* form.

Theorem. *Let f be a continuous map of a CW complex X into a CW complex Y such that the restriction $f|_A$ is cellular for some CW subcomplex A of X. Then there exists a cellular map $g: X \to Y$ such that $g|_A = f|_A$, and, moreover, g is A-homotopic to f.*

The expression "g is A-homotopic to f" (in formulas, $g \sim_A f$) means that there is a homotopy h_t between g and f which is fixed on A; that is, $f_t(x)$ does not depend on t for every $x \in A$. It is clear that if $g \sim_A f$, then $g|_A = f|_A$. Certainly, $g \sim_A f$ implies $g \sim f$, but not vice versa. For example, the maps $f, g: I \to S^1$, where f is the winding of the segment about the circle mapping both endpoints into the same point of the circle and g is a constant map, are homotopic, but not $(0 \cup 1)$-homotopic (strictly speaking, we will prove this only in Lecture 6).

Proof of Theorem. Assume that the map f has already been made cellular not only on all cells from A, but also on all cells from X of dimensions less than p. Take a p-dimensional cell $e^p \subset X - A$. Its image $f(e^p)$ has a nonempty intersection with only a finite set of cells of Y [this follows from the compactness of $f(\overline{e^p})$—see Exercise 3]. Of these cells of Y, choose a cell of a maximal dimension, say, ϵ^q, $\dim \epsilon^q = q$. If $q \le p$, then we do not need to do anything with the cell e^p. If, however, $q > p$, we will need the following lemma.

Free-Point Lemma. *Let U be an open subset of $\mathbb{R}^p$ and $\varphi: U \to \mathrm{Int}\, D^q$ be such a continuous map that the set $V = \varphi^{-1}(d^q) \subset U$ where d^q is some closed ball in $\mathrm{Int}\, D^q$ is compact. If $q > p$, then there exists a continuous map $\psi: U \to \mathrm{Int}\, D^q$ coinciding with φ in the complement of V and such that its image does not cover the whole ball d^q.*

We will postpone the proof of this lemma (and a discussion of its geometric meaning) until the next section. For now, we restrict ourselves to the following obvious remark. The map ψ is automatically $(U-V)$-homotopic to φ: It is sufficient to take the "straight" homotopy joining φ and ψ when, for every $u \in U$, the point $\varphi(u)$ is moving to $\psi(u)$ at a constant speed along a straight interval joining $\varphi(u)$ and $\psi(u)$.

Now, let us finish the proof of the theorem. The free-point lemma implies that *the restriction $f_{A\cup X^{p-1}\cup e^p}$ is $(A \cup X^{p-1})$-homotopic to a map $f'\colon A \cup X^{p-1} \cup e^p \to Y$ such that $f'(e^p)$ has nonempty intersections with the same cells as $f(e^p)$, but $f'(e^p)$ does not cover the whole cell ϵ^q*. Indeed, let $h\colon D^p \to X$ and $k\colon D^q \to Y$ be characteristic maps corresponding to the cells e^p and ϵ^q. Let $U = h^{-1}(f^{-1}(\epsilon^q) \cap e^q)$ and define a map $\varphi\colon U \to \operatorname{Int} D^q$ as a composition

$$\begin{array}{ccccccc} u & \overset{h}{\longmapsto} & x & \overset{f}{\longmapsto} & y & \overset{k^{-1}}{\longmapsto} & v = \varphi(u) \\ \Cap & & \Cap & & \Cap & & \Cap \\ U & & e^p \cup f^{-1}(e^q) & & e^q & & \text{Int } D^q \end{array}$$

Denote as d^q a closed concentric subball of the ball D^q. The set $V = \varphi^{-1}(d^q)$ is compact (because it is a closed subset of a closed ball D^p). Let $\psi\colon U \to \operatorname{Int} D^q$ be a map provided by the free-point lemma. We define the map f' as coinciding with f in the complement of $h(U)$ and as the composition

$$\begin{array}{ccccccc} x & \overset{h^{-1}}{\longmapsto} & u & \overset{\psi}{\longmapsto} & v & \overset{k}{\longmapsto} & y = f'(u) \\ \Cap & & \Cap & & \Cap & & \Cap \\ h(U) & & U & & \text{Int } D^q & & e^q \subset Y \end{array}$$

in $h(U)$. It is clear that the map f' is continuous [it coincides with f on the "buffer" set $h(U-V)$] and $(A \cup X^{p-1})$-homotopic [actually, even $(A \cup X^{p-1} \cup (e^p - h(V)))$-homotopic] to $f|_{A\cup X^{p-1}\cup e^p}$ [because $\varphi \sim_{(U-V)} \psi$]. It is also clear that $f'(e^p)$ does not cover ε^q.

It is very easy now to complete the proof. First, by Borsuk's theorem, we can extend our homotopy fixed on $A \cup X^{p-1}$ between $f|_{A\cup X^{p-1}\cup e^p}$ and f' to the whole space X, which lets us assume that the map f' with all necessary properties is defined on the whole space X. After that, we take a point $y_0 \in \epsilon^q$, not in $f(e^p)$, and apply to $f'|_{e^p}$ a "radial homotopy": If $x \in e^p - f^{-1}(\varepsilon^q)$, then $f'(x)$ does not move, but if $f'(x) \in \epsilon^q$, then $f'(x)$ is moving, at a constant speed, along a straight path going from y_0 through $f'(x)$ to the boundary of ϵ^q [more precisely, along the k-image of a straight interval in D^q starting at $k^{-1}(y_0)$ and going through $k^{-1}(f'(x))$ to the boundary sphere S^{q-1}]. We extend this homotopy to a homotopy of $f'|_A \cup X^{p-1} \cup e^p$ (fixed in the complement of e^p), and then, using Borsuk's theorem, to a homotopy of the whole map $f'\colon X \to Y$. In this way, we reduce the number of q-dimensional cells hit by $f'(e^p)$ by one, and, repeating this procedure a necessary amount of times,